

Homology and dynamics in quasi-isometric rigidity of once-punctured mapping class groups

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Abstract

In these lecture notes, we combine recent homological methods of Kevin Whyte with older dynamical methods developed by Benson Farb and myself, to obtain a new quasi-isometric rigidity theorem for the mapping class group $\mathcal{MCG}(S_g^1)$ of a once punctured surface S_g^1 : if K is a finitely generated group quasi-isometric to $\mathcal{MCG}(S_g^1)$ then there is a homomorphism $K \rightarrow \mathcal{MCG}(S_g^1)$ with finite kernel and finite index image. This theorem is joint with Kevin Whyte.

Gromov proposed the program of classifying finitely generated groups according to their large scale geometric behavior. The goal of this paper is to combine recent homological methods of Kevin Whyte with older dynamical methods developed by Benson Farb and myself, to obtain a new quasi-isometric rigidity theorem for mapping class groups of once punctured surfaces:

Theorem 1 (Mosher-Whyte). *If S_g^1 is an oriented, once-punctured surface of genus $g \geq 2$ with mapping class group $\mathcal{MCG}(S_g^1)$, and if K is a finitely generated group quasi-isometric to $\mathcal{MCG}(S_g^1)$, then there exists a homomorphism $K \rightarrow \mathcal{MCG}(S_g^1)$ with finite kernel and finite index image.*

This theorem will be restated later with a more quantitatively precise conclusion; see Theorem 9.

Whyte is also able to apply his techniques to obtain a strong quasi-isometric rigidity theorem for the group $\mathbf{Z}^n \rtimes \mathrm{GL}(n, \mathbf{Z})$, which we will not state here.

Our theorem about $\mathcal{MCG}(S_g^1)$, answers a special case of:

Conjecture 2. *If S is a nonexceptional surface of finite type then for any finitely generated group K quasi-isometric to $\mathcal{MCG}(S)$ there exists a homomorphism $K \rightarrow \mathcal{MCG}(S)$ with finite kernel and finite index image.*

The exceptional surfaces that should be ruled out include several for which we already have quasi-isometric rigidity theorems of a different type: the sphere with ≤ 3 punctures whose mapping class groups are finite; the once-punctured torus and the four punctured sphere whose mapping class groups are commensurable to a free group of rank ≥ 2 . Probably the techniques used for the once-punctured case may not be too useful in the general case.

The theorem about $\mathcal{MCG}(S_g^1)$, and Whyte's results about $\mathbf{Z}^n \rtimes \mathrm{GL}(n, \mathbf{Z})$, are both about "universal extension" groups of certain PD(n) groups: $\mathcal{MCG}(S_g^1) \approx \mathrm{Aut}(\pi_1 S_g)$ is the universal extension of the PD(2) group $\pi_1 S_g$; and $\mathbf{Z}^n \rtimes \mathrm{GL}(n, \mathbf{Z})$ is the universal extension of the PD(n) group \mathbf{Z}^n . If one wishes to pursue quasi-isometric rigidity for the group $\mathrm{Aut}(F_n)$, where F_n is the free group of rank ≥ 2 , noting that $\mathrm{Aut}(F_n)$ is the universal extension of F_n , the difficulty is that the homological techniques we shall use do not apply: the Poincaré duality groups $\pi_1 S_g$ and \mathbf{Z}^n each have a fundamental class in uniformly finite homology, which F_n does not have.

Contents: This paper is based on L^AT_EX slides that were prepared for lectures given at the LMS Durham Symposium on Geometry and Cohomology in Group Theory, July 2003. Here is an outline of the paper, based approximately on my four lectures at the conference:

1. Survey of results and techniques in quasi-isometric rigidity.
2. Whyte's techniques: uniformly finite homology applied to extension groups.
3. Surface group extensions and Mess subgroups.
4. Dynamical techniques: extensions of surface groups by pseudo-Anosov homeomorphisms.

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1 Survey of results and techniques in QI-rigidity

A map $f: X \rightarrow Y$ of metric spaces is a *quasi-isometric embedding* if $\exists K \geq 1, C \geq 0$ such that

$$\frac{1}{K} \cdot d_X(x, y) - C \leq d_Y(fx, fy) \leq K \cdot d_X(x, y) + C$$

A *coarse inverse* for f is a quasi-isometry $\bar{f}: Y \rightarrow X$ s.t.

$$d_{\mathrm{sup}}(\bar{f} \circ f, \mathrm{Id}_X), \quad d_{\mathrm{sup}}(f \circ \bar{f}, \mathrm{Id}_Y) < \infty$$

A coarse inverse exists if and only if $\exists C' \geq 0$ such that $\forall y \in Y \exists x \in X$ such that

$$d_Y(fx, y) \leq C'$$

If this happens then $f: X \rightarrow Y$ is a *quasi-isometry*, and X, Y are *quasi-isometric* metric spaces. We will use the abbreviation “QI” for “quasi-isometric”.

Given a finitely generated group G , a *model space* for G is a metric space X on which G acts by isometries such that:

- X is *proper*, meaning that closed balls are compact.
- X is *geodesic*, meaning that any $x, y \in X$ are connected by a rectifiable path γ such that $\text{Length}(\gamma) = d(x, y)$.
- The action is properly discontinuous and cobounded.

Examples of model spaces:

- The Cayley graph of G with respect to a finite generating set.
- $X = \tilde{Y}$ where Y is a compact, connected Riemannian manifold or piecewise Riemannian cell complex, and $G = \pi_1 Y$.

Fact: If X, Y are two model spaces for G then X, Y are quasi-isometric. Also, any model space is quasi-isometric to G with its word metric.

As a consequence, a finitely generated group G has a notion of geometry that is well-defined up to quasi-isometry, namely the geometry of any model space, or of G itself with a word metric.

Definition: Two finitely generated groups are *quasi-isometric* if, when equipped with their word metrics, they become quasi-isometric as metric spaces; equivalently, their Cayley graphs are quasi-isometric.

Notation: Given \mathcal{G} a collection of finitely generated groups, let $\langle \mathcal{G} \rangle = \{\text{all groups quasi-isometric to some group in } \mathcal{G}\}$. More generally, given \mathcal{X} a collection of metric spaces, let $\langle \mathcal{X} \rangle = \{\text{all groups quasi-isometric to some metric space in } \mathcal{X}\}$.

Examples of QI-rigidity theorems. To reformulate Gromov’s program in a practical way: given a collection of metric spaces \mathcal{X} , describe the collection of groups $\langle \mathcal{X} \rangle$, preferably in simple algebraic or geometric terms that do not invoke the concept of quasi-isometry. Also, describe all of the quasi-isometry classes within $\langle \mathcal{X} \rangle$. In particular, identify interesting classes of groups \mathcal{G} that are *QI-rigid*, meaning $\mathcal{G} = \langle \mathcal{G} \rangle$. There are many theorems describing interesting QI-rigid classes of groups, proved using an incredibly broad range of mathematical tools.

Example: Gromov’s polynomial growth theorem [Gro81] implies

Theorem 3. *The class of virtually nilpotent groups is quasi-isometrically rigid. The class of virtually abelian groups is quasi-isometrically rigid, with one QI-class for each rank.*

Within the class of virtually nilpotent groups, there are many interesting QI-invariants:

- The Hirsch rank is a QI-invariant.
- The sequence of ranks of the abelian subquotients is a (finer) QI-invariant.
- There is an even finer QI-invariant of a virtually nilpotent Lie group G : Pansu proved that the asymptotic cone of G is a graded Lie group, whose associated graded Lie algebra is a quasi-isometry invariant that subsumes the previous invariants [Pan83].
- This is still not the end of the story: recently Yehuda Shalom produced two finitely generated nilpotent groups which are not quasi-isometric but whose associated graded Lie algebras are isomorphic [Sha02].

The full QI-classification of virtually nilpotent groups remains unknown.

Example: Stallings’ ends theorem [Sta68] implies

Theorem 4. *The class of groups which splits over a finite group is quasi-isometrically rigid. For each $n \geq 2$, the class $\langle F_n \rangle$ consists of all groups that are virtually free of rank ≥ 2 .*

By work of Papasoglu and Whyte [PW02], combined with Dunwoody’s accessibility theorem [Dun85], the QI classification of finitely presented groups that split over a finite group is completely reduced to the QI classification of one ended groups.

Example: Sullivan proved [Sul81] that any uniformly quasiconformal action on S^2 is quasiconformally conjugate to a conformal action. This implies:

Theorem 5 (Sullivan–Gromov). *$\langle \mathbf{H}^3 \rangle$ consists of all groups H for which there exists a homomorphism $H \rightarrow \text{Isom}(\mathbf{H}^3)$ with finite kernel and whose image is a cocompact lattice.*

This theorem is prototypical of a broad range of QI-rigidity theorems, including our theorem about $\mathcal{MCG}(S_g^1)$. However, the conclusion of our theorem should be contrasted with the Sullivan–Gromov theorem: the latter gives only a “topological” characterization of $\langle \mathbf{H}^3 \rangle$, which does not serve to give us an effective list of those groups in $\langle \mathbf{H}^3 \rangle$. There is still no effective listing of the cocompact lattices acting on \mathbf{H}^3 . The conclusion of our theorem gives an “algebraic” characterization of $\langle \mathcal{MCG}(S_g^1) \rangle$, allowing an effective listing.

Example: Rich Schwartz proved a strong quasi-isometric rigidity theorem for noncocompact lattices in $\text{Isom}(\mathbf{H}^3)$ [Sch96]. To state the theorem we need some definitions.

The commensurator group: Given two groups G, H , a *commensuration* from G to H is an isomorphism from a finite index subgroup of G to a finite index subgroup of H . Two commensurations are *equivalent* if they agree upon restriction to another finite index subgroup. The *commensurator group* $\text{Comm}(G)$ is the set of self-commensurations of G up to equivalence, with the following group law: given commensurations $\phi: A \rightarrow B, \psi: C \rightarrow D$ restrict the range of ϕ and the domain of ψ to the finite index subgroup $B \cap C$, and then compose $\psi \circ \phi$.

The left action of G on itself by conjugation induces a homomorphism $G \rightarrow \text{Comm}(G)$, whose kernel is the *virtual center* of G , consisting of all elements $g \in G$ such that the centralizer of g has finite index in G .

Two groups G, H are *abstractly commensurable* if there exists a commensuration from G to H . Any abstract commensuration from G to H induces an isomorphism from $\text{Comm}(G)$ to $\text{Comm}(H)$.

Theorem 6 (Schwartz). *If G is a noncocompact, nonarithmetic lattice in $\text{Isom}(\mathbf{H}^3)$, then $\langle G \rangle$ consists of those finitely generated groups H which are abstractly commensurable to G . More precisely, the homomorphism $G \rightarrow \text{Comm}(G)$ is an injection with finite index image, and $\langle G \rangle$ consists of those finitely generated groups H for which there exists a homomorphism $H' \rightarrow \text{Comm}(G)$ with finite kernel and finite index image.*

This theorem gives a very precise and effective enumeration of $\langle \mathcal{G} \rangle$, similar to the conclusion of our main theorems. Schwartz' theorem also can be formulated in the arithmetic case, although there the homomorphism $G \rightarrow \text{Comm}(G)$ has infinite index image.

The general techniques of Sullivan-Gromov theorem, and of Schwartz' theorem give models for the proof of our main theorem, as we now explain.

Technique: the quasi-isometry group of a group. Consider a metric space X , for example a model space for a finitely generated group. Let $\widehat{\text{QI}}(X)$ be the set of self quasi-isometries of X , equipped with the operation of composition. Define an equivalence relation on $\widehat{\text{QI}}(X)$, where $f \sim g$ if $d_{\text{sup}}(f, g) = \sup\{fx, gx\} < \infty$. Composition descends to a group operation on the set of equivalence classes, giving a group

$$\text{QI}(X) = \text{the quasi-isometry group of } X$$

Notation: let $[f]$ denote the equivalence class of f in $\text{QI}(X)$. Note that $[f]^{-1} = [\bar{f}]$ for any coarse inverse \bar{f} to f .

For any quasi-isometry $f: X \rightarrow Y$ we obtain an isomorphism $\text{ad}_f: \text{QI}(X) \rightarrow \text{QI}(Y)$ defined by $\text{ad}_f[g] = [f \circ g \circ \bar{f}]$, where \bar{f} is any coarse inverse for f .

It follows that if G is a finitely generated group then the *quasi-isometry group* of G is well-defined up to isomorphism by taking it to be $\text{QI}(X)$ for any model space X of G .

The group $\text{QI}(G)$ is an important quasi-isometry invariant of a group G , and it is often important to be able to compute it. Here are some properties of $\text{QI}(G)$, followed a little later by some examples of computations.

The left action of G on itself by multiplication, defined by $L_g(h) = gh$, induces a homomorphism $G \rightarrow \text{QI}(G)$ whose kernel is the virtual center. Also, the left action of G on itself by conjugation, defined by $C_g(h) = ghg^{-1}$, also defines a homomorphism $G \rightarrow \text{Comm}(G)$. These two homomorphisms are identical, because $d_{\text{sup}}(L_g, C_g)$ equals the word length of g .

Every commensuration defines a natural quasi-isometry of G , well defined in $\text{QI}(G)$ up to equivalence of commensurations, thereby defining a homomorphism $\text{Comm}(G) \rightarrow \text{QI}(G)$. The homomorphism $G \rightarrow \text{QI}(G)$ factors as

$$G \rightarrow \text{Comm}(G) \rightarrow \text{QI}(G)$$

Technique: quasi-actions Let G be a finitely generated group, X a model space for G , and H a finitely generated group quasi-isometric to G . Fix a quasi-isometry $\Phi: H \rightarrow X$ and a coarse inverse $\bar{\Phi}: X \rightarrow H$. Define $A: H \rightarrow \widehat{\text{QI}}(X)$ by the formula

$$A(h) = \Phi \circ L_h \circ \bar{\Phi}$$

This map has the following properties:

A is a quasi-action: There exists constants $K \geq 1$, $C \geq 0$ such that

- The maps $A(h)$ are K, C quasi-isometries for all $h \in H$
- $d_{\text{sup}}(A(hh'), A(h) \circ A(h')) \leq C$ for all $h, h' \in H$
- $d_{\text{sup}}(A(\text{Id}), \text{Id}) \leq C$

and so we obtain a homomorphism $A: H \rightarrow \text{QI}(X)$.

A is proper: $\forall r \geq 0 \exists n$ such that if $B, B' \subset X$ have diameter $\leq r$ then

$$|\{h \in H \mid (A(h) \cdot B) \cap B' \neq \emptyset\}| \leq n$$

A is cobounded: $\exists s \geq 0$ such that $\forall x, y \in X \exists h \in H$ such that $d(A(h) \cdot x, y) \leq s$.

Given a group G and a model space X , a common strategy in investigating quasi-isometric rigidity of G is: compute $\text{QI}(X)$; and then describe those homomorphisms $H \rightarrow \text{QI}(X)$ arising from quasi-actions, called “uniform” homomorphism. If necessary, restrict to proper, cobounded quasi-actions. Try to “straighten” any such quasi-action.

Examples of QI-rigidity. Here are some examples of how this strategy is carried out, taken from the above examples:

Proof sketch for the Sullivan–Gromov rigidity theorem. For groups in the quasi-isometry class of \mathbf{H}^3 , the boundary is $\partial\mathbf{H}^3 = S^2$.

First, one calculates $\text{QI}(\mathbf{H}^3) = \text{QC}(S^2)$, the group of quasi-conformal homeomorphisms of S^2 ; this is a classical result in quasiconformal geometry.

Second, the isometry group $\text{Isom}(\mathbf{H}^3) = \text{Conf}(S^2)$ is a uniform subgroup of $\text{QC}(S^2)$, and one proves that every uniform subgroup can be conjugated into $\text{Conf}(S^2)$. In other words, every quasi-action on \mathbf{H}^3 is quasiconjugate to an action. This result, due to Sullivan, is the heart of the proof.

The properties of “properness” and “coboundedness” are invariant under quasiconjugacy. It follows that if H is a finitely generated group quasi-isometric to \mathbf{H}^3 then H has a proper, cobounded action on \mathbf{H}^3 . In other words, there is a homomorphism $H \rightarrow \text{Isom}(\mathbf{H}^3)$ with finite kernel and discrete, cocompact image. \diamond

By contrast we now give:

Proof sketch for the Schwartz rigidity theorem. Let G be a noncocompact lattice in \mathbf{H}^3 .

The heart of the proof is essentially a calculation

$$\text{QI}(G) \approx \text{Comm}(G)$$

This calculation holds in both the arithmetic case and the nonarithmetic case, the difference being that the induced map $G \rightarrow \text{Comm}(G)$ has finite index image if and only if G is nonarithmetic. Assuming this to be the case, it follows that the homomorphism $\text{Comm}(G) \rightarrow \text{QI}(G)$ is an isomorphism and that the map $G \rightarrow \text{Comm}(G) \approx \text{QI}(G)$ is an injection with finite index image.

Schwartz’ proof is actually a bit more quantitative, as follows. If G is nonarithmetic then there exists an embedding

$$\text{Comm}(G) \hookrightarrow \text{Isom}(\mathbf{H}^3)$$

whose image Γ is a noncocompact lattice containing G with finite index, so that the injection $G \rightarrow \text{Comm}(G)$ agrees with the inclusion $G \hookrightarrow \Gamma$. The hard part of Schwartz’ proof is to show the following:

- $\forall K \geq 1, C \geq 0 \exists A \geq 0$ such that if $\Phi: G \rightarrow G$ is a K, C quasi-isometry then there exists $\gamma \in \Gamma$ such that

$$d_{\text{sup}}(\Phi, L_\gamma) \leq A$$

To be more precise, the sup distance on the left is a comparison of two different functions from G into Γ , one being $G \xrightarrow{\Phi} G \hookrightarrow \Gamma$, and the other being $G \hookrightarrow \Gamma \xrightarrow{L_\gamma} \Gamma$.

Noting that any quasi-isometry of G extends to the finite index supergroup Γ , and that $\text{QI}(G) \approx \text{QI}(\Gamma)$, we can abstract this discussion as follows.

Consider a finitely generated group Γ , and suppose that the following holds:

Strong QI-rigidity: $\forall K \geq 1, C \geq 0 \exists A \geq 0$ such that if $\Phi: \Gamma \rightarrow \Gamma$ is a K, C quasi-isometry then there exists $\gamma \in \Gamma$ such that

$$d_{\text{sup}}(\Phi, L_\gamma) \leq A$$

This property, coupled with triviality of the virtual center (true for lattices in $\text{Isom}(\mathbf{H}^3)$ as well as for $\mathcal{MCG}(S_g^1)$), immediately imply that the homomorphism $\Gamma \rightarrow \text{QI}(\Gamma)$ is an isomorphism.

To complete the proof of Schwartz' Theorem, we now apply the following fact:

Proposition 7. *If Γ is a strongly QI-rigid group whose virtual center is trivial, then for any finitely generated group H quasi-isometric to Γ there exists a homomorphism $H \rightarrow \Gamma$ with finite kernel and finite index image.*

Proof. As explained earlier, the left action of H on itself by translation can be quasiconjugated to a proper, cobounded quasi-action of H on Γ , which induces a homomorphism $\phi: H \rightarrow \text{QI}(\Gamma) = \Gamma$.

Let $K \geq 1, C \geq 0$ be uniform constants for the quasi-action of H on Γ .

Applying strong QI-rigidity of Γ , we obtain a constant A such that the (quasi-)action of each $h \in H$ on Γ is within sup distance A of left multiplication by $\phi(h)$. It immediately follows that the kernel of ϕ is finite, because the quasi-action of H is proper and so there are only finitely many elements $h \in H$ for which $\phi(h)$ is within distance A of the identity on Γ .

It also follows that the image of ϕ has finite index, because the quasi-action of H on Γ is cobounded, whereas the left action on Γ of any infinite index subgroup of Γ is not cobounded. \diamond

This completes the proof of Schwartz' Theorem. \diamond

This proof immediately yields an interesting corollary:

Corollary 8. *If Γ is strongly QI-rigid with trivial virtual center, then every commensuration of Γ is the restriction of an inner automorphism of Γ .*

Now I can give the more quantitative statement of the main theorem about $\mathcal{MCG}(S_g^1)$:

Theorem 9 (Mosher-Whyte). *The group $\mathcal{MCG}(S_g^1)$ is strongly QI-rigid: for all $K \geq 1$, $C \geq 0$ there exists $A \geq 0$ such that for any K, C quasi-isometry $\Phi: \mathcal{MCG}(S_g^1) \rightarrow \mathcal{MCG}(S_g^1)$ there exists $\gamma \in \mathcal{MCG}(S_g^1)$ for which $d_{\text{sup}}(\Phi, L_\gamma) < A$.*

As an application, we get a new proof of a result of Ivanov:

Corollary 10 (Ivanov). *The injection $\mathcal{MCG}(S_g^1) \rightarrow \text{Comm}(S_g^1)$ is an isomorphism, that is, every commensuration of $\mathcal{MCG}(S_g^1)$ is the restriction of an inner automorphism.*

2 Fiber preserving quasi-isometries

In this section we explain Kevin Whyte's methods for using uniformly finite homology classes and their supports to investigate quasi-isometric rigidity problems for certain fiber bundles. Bruce Kleiner also outlined, at the AMS Ann Arbor conference in 2000, how to use support sets to study quasi-isometric rigidity, using a coarse version of the Künneth formula applied to fiber bundles.

Suppose one wants to investigate quasi-isometric rigidity for the fundamental group of a graph of groups where each vertex and edge group is (virtually) π_1 of an aspherical n -manifold for fixed integer $n \geq 0$. Let us focus on the example $F_2 \times \mathbf{Z}^n$.

To start the proof, pick a nice model space for $F_2 \times \mathbf{Z}^n$, namely, $T \times \mathbf{R}^n$ where T is a Cayley tree of F_2 . As described earlier, the technique will be to study quasi-actions on $T \times \mathbf{R}^n$. The first step, carried out by Farb and myself, is to prove that each quasi-isometry of $T \times \mathbf{R}^n$ coarsely preserves the \mathbf{R}^n fibers:

Theorem 11 ([FM00]). *$\forall K, C \exists A$ such that if $\Phi: T \times \mathbf{R}^n \rightarrow T \times \mathbf{R}^n$ is a K, C quasi-isometry, then $\forall t \in T \exists t' \in T$ such that $d_{\mathcal{H}}(\Phi(t \times \mathbf{R}^n), t' \times \mathbf{R}^n) < A$.*

The notation $d_{\mathcal{H}}(\cdot, \cdot)$ means Hausdorff distance between subsets of a metric space.

More generally, this theorem is true when the product $T \times \mathbf{R}^n$ is replaced by a “coarse fibration” over a tree whose fiber is a uniformly contractible manifold, or even more generally by the Bass-Serre tree of spaces that arise from a finite graph of groups whose vertex and edge groups are coarse PD(n) groups of fixed dimension n [MSW03].

In this section we shall give Kevin Whyte's proof of Theorem 11. This proof also applies to situations where the base space T of the fibration is replaced by certain higher dimensional complexes, for example:

- Thick buildings.
- The model space for $\mathcal{MCG}(S_g)$, over which a model space for $\mathcal{MCG}(S_g^1)$ fibers, with fiber \mathbf{H}^2 .
- A model space for $\text{SL}(n, \mathbf{Z})$, over which a model space for $\mathbf{Z}^n \rtimes \text{GL}(n, \mathbf{Z})$ fibers, with fiber \mathbf{R}^n .

For the example $T \times \mathbf{R}^n$, the idea of the proof is that a subset of the form $(\text{line in } T) \times \mathbf{R}^n \approx \mathbf{R}^{n+1}$ is the support of a “top dimensional uniformly finite homology class”. A quasi-isometry of $T \times \mathbf{R}^n$ acts on such classes, coarsely preserving their supports. Each fiber is the intersection of some finite number of these supports, and so the fibers are preserved.

In this proof it is necessary that the fiber be a uniformly contractible *manifold*, on which there is a “uniformly finite” fundamental class of full support. (For those who live in outer space, that’s why the proof does not apply to the extension $1 \rightarrow F_n \rightarrow \text{Aut}(F_n) \rightarrow \text{Out}(F_n) \rightarrow 1$, whose fiber F_n is not a manifold and does not have a uniformly finite fundamental class of full support).

In order to make this proof rigorous, we have to discuss:

1. Uniformly finite homology
2. Top dimensional supports
3. Application to fiber bundles

2.1 Uniformly finite homology

Let X be a simplicial complex. Fix a geodesic metric in which each simplex is a regular Euclidean simplex with side length 1. We say that X is *uniformly locally finite* or ULF if $\exists A \geq 0$ such that the link of each simplex contains at most A simplices. We say that X is *uniformly contractible* or UC if $\forall r > 0 \exists s(r) \geq 0$ s.t. each subset $A \subset X$ with $\text{diam}(A) \leq r$ is contractible to a point inside $N_s(A)$. The function $s(r)$ is called a *gauge* of uniform contractibility.

For example, any tree is uniformly contractible. Also, if X is a contractible simplicial complex, and if there is a cocompact, simplicial group action on X (for example if X is the universal cover of a compact, aspherical simplicial complex), the X is UC and ULF.

Simplicial uniformly finite homology. If X is ULF, define $H_n^{\text{suf}}(X)$: a chain in $C_n^{\text{suf}}(X)$ is a uniformly bounded assignments of integers to n -simplices. Since X is ULF, the boundary map $\partial: C_n^{\text{suf}}(X) \rightarrow C_{n-1}^{\text{suf}}(X)$ is defined, and clearly $\partial\partial = 0$.

Theorem 12. $H_*^{\text{suf}}(X)$ is a quasi-isometry invariant among UC, ULF simplicial complexes.

Proof. We prove the theorem by defining *uniformly finite homology* $H_*^{\text{uf}}(X)$ which is a large scale version of simplicial uniformly finite homology $H_*^{\text{suf}}(X)$, proving that $H_*^{\text{uf}}(X)$ and $H_*^{\text{suf}}(X)$ are isomorphic for UC, ULF simplicial complexes, and then proving that $H_*^{\text{uf}}(X)$ is a QI-invariant.

Step 1: Uniformly finite homology. Define the n th Rips complex $R^n(X)$, with one k -simplex for each ordered $k+1$ -tuple of vertices with diameter $\leq n$. Note that $R^1(X) = X$.

Since X is ULF, it follows that $R^n(X)$ is ULF, and the sequence of inclusions

$$X = R^1(X) \subset R^2(X) \subset \dots$$

therefore induces homomorphisms

$$H_*^{\text{suf}}(R^1(X)) \rightarrow H_*^{\text{suf}}(R^2(X)) \rightarrow H_*^{\text{suf}}(R^3(X)) \dots$$

Define the uniformly finite homology to be the direct limit

$$H_*^{\text{uf}}(X) = \lim_{k \rightarrow \infty} H_*^{\text{suf}}(R^k(X))$$

We can also describe $H_*^{\text{uf}}(X)$ as the homology of a chain complex. We have a direct system

$$C_*^{\text{suf}}(R^1(X)) \rightarrow C_*^{\text{suf}}(R^2(X)) \rightarrow C_*^{\text{suf}}(R^3(X)) \rightarrow \dots$$

so we can take the direct limit

$$C_*^{\text{uf}}(X) = \lim_{k \rightarrow \infty} C_*^{\text{suf}}(R^k(X))$$

The boundary homomorphism is defined, and the homology of this chain complex is canonically isomorphic to $H_*^{\text{uf}}(X)$.

Step 2: $H_*^{\text{uf}}(X) = H_*^{\text{suf}}(X)$.

The identity map $i: X \rightarrow X = R^1(X)$ induces a chain map $i: C_*^{\text{suf}}(X) \rightarrow C_*^{\text{uf}}(X)$. Using that X is UC, we'll define a chain map

$$j: C_*^{\text{uf}}(X) \rightarrow C_*^{\text{suf}}(X)$$

which is a uniform chain homotopy inverse to the inclusion $C_*^{\text{suf}}(X) \rightarrow C_*^{\text{uf}}(X)$.

The idea for defining j is: connect the dots.

Consider a 1-simplex in $C_1^{\text{uf}}(X)$, which means a 1-simplex σ in $R^k(X)$ for some k , which means $\sigma = (u, v)$ with $d(u, v) \leq k$. Connecting the dots, we get a 1-chain $j(\sigma)$ in X with boundary $v - u$, consisting of at most k 1-simplices. Given now a simplicial uniformly finite 1-chain $\sum a_\sigma \sigma$ in $R^k(X)$, the infinite sum

$$j(\sum a_\sigma \sigma) = \sum a_\sigma j(\sigma)$$

is defined because it is locally finite: for each simplex τ in X there are finitely many terms of the sum $\sum a_\sigma j(\sigma)$ which assign a nonzero coefficient to τ . This finishes the definition of $j: C_1^{\text{uf}}(X) \rightarrow C_1^{\text{suf}}(X)$.

Next consider a 2-simplex σ in $C_2^{\text{uf}}(X)$, which means $\sigma = (u, v, w)$, where u, v, w have pairwise distances at most k . Now connect the 2-dimensional dots: the 1-chain

$$j(u, v) + j(v, w) + j(w, u)$$

is a cycle. Its support is a subset of diameter at most $3k/2$, and so

$$j(u, v) + j(v, w) + j(w, u) = \partial j(\sigma)$$

for some 2-chain $j(\sigma)$ supported on a subset of diameter at most $s(3k/2)$, where s is a gauge of uniform contractibility. More generally, the boundary of a simplicially uniformly finite 2-chain in $R^2(X)$, is again defined as a locally finite infinite sum. This finishes the definition of $j: C_2^{\text{uf}}(X) \rightarrow C_2^{\text{suf}}(X)$, and the chain map condition is obvious.

Now continue the definition of the chain map j by induction, using connect-the-dots.

Similarly, using connect the dots and induction, we can construct a chain homotopy between identity and ji , and similarly for ij . This finishes Step 2.

Step 3: QI-invariance of uniformly finite homology. Consider a quasi-isometry $\Phi: X \rightarrow Y$ with coarse inverse $\bar{\Phi}: Y \rightarrow X$, both K, C quasi-isometries, and C -coarse inverses of each other. Moving a bounded distance, may assume $\Phi, \bar{\Phi}$ take vertices to vertices.

If $d(u, v) = 1$ then $d(\Phi u, \Phi v) \leq p = K + C$. We therefore obtain an induced simplicial map $X = R^1(X) \rightarrow R^p(X)$, inducing a chain map

$$\Phi_{\#}: C_*^{\text{suf}}(R^1(X)) \rightarrow C_*^{\text{suf}}(R^p(X))$$

In the backwards direction, if $d(u', v') \leq p$ in Y , then $d(\bar{\Phi}u', \bar{\Phi}v') \leq p' = Kp + C$, and so we get an induced simplicial map $R^p(Y) \rightarrow R^{p'}(X)$ inducing a chain map

$$\bar{\Phi}_{\#}: C_*^{\text{suf}}(R^p(Y)) \rightarrow C_*^{\text{suf}}(R^{p'}(X))$$

The composition $\bar{\Phi} \circ \Phi$ induces a chain map

$$\bar{\Phi}_{\#} \circ \Phi_{\#}: C_*^{\text{suf}}(R^1(X)) \rightarrow C_*^{\text{suf}}(R^{p'}(X))$$

but $\bar{\Phi} \circ \Phi$ is C -close to the identity map on vertices of X , and so a connect-the-dots argument shows that this chain map is chain homotopic to the inclusion map.

A similar argument applies to the composition $\Phi \circ \bar{\Phi}$.

This finishes the proof that H_*^{suf} is a QI invariant. ◇

2.2 Top dimensional supports.

Suppose now that X is a UC, ULF simplicial complex of dimension d . There are no simplices of dimension $d + 1$, and so each class $c \in H_d^{\text{uf}}(X)$ is represented by a unique d -cycle in $C_d^{\text{suf}}(X)$, also denoted c . Its support $\text{supp}(c)$ is therefore a well-defined subset of X .

Proposition 13. *With X as above, every quasi-isometry of X coarsely respects supports of classes in $H_d^{\text{uf}}(X)$. More precisely: $\forall K, C \exists A$ such that if $\Phi: X \rightarrow X$ is a K, C quasi-isometry, and if $c \in H_d^{\text{uf}}(X)$, then*

$$d_{\mathcal{H}}(\Phi(\text{supp}(c)), \text{supp}(\Phi_*(c))) < A$$

Most QI-rigidity theorems have a similar step: find some collection \mathcal{C} of objects in the model space which are coarsely respected by quasi-isometries: $\forall K, C \exists A$ such that for each K, C quasi-isometry $\Phi: X \rightarrow X$, and for each object $c \in \mathcal{C}$ there exists an object $c' \in \mathcal{C}$ such that

$$d_{\mathcal{H}}(\Phi(c), c') < A$$

Proof. Moving Φ a bounded distance, we may assume that Φ takes vertices to vertices. We get an induced chain map

$$\Phi_{\#}: C^{\text{suf}}(X) \rightarrow C^{\text{suf}}(R^p(X))$$

Note first that for all $c \in C^{\text{suf}}(X)$, the subset $\text{supp}(\Phi_{\#}(c))$ is contained in a uniformly bounded neighborhood of $\Phi_{\#}(\text{supp}(c))$, where the support of a chain in $C^{\text{suf}}(R^p(X))$ is simply the set of vertices occurring among the summands in the chain. In other words, $\Phi_{\#}$ induces coarse inclusion of supports.

Compose with the connect-the-dots map

$$C^{\text{suf}}(R^p(X)) \rightarrow C^{\text{suf}}(X)$$

which also induces coarse inclusion of supports. By composition we obtain an induced map

$$\Phi_{\#\#}: C^{\text{suf}}(X) \rightarrow C^{\text{suf}}(X)$$

which also induces coarse inclusion of supports. Since top dimensional supports are unique, it follows that

$$\Phi(\text{supp}(c)) \subset N_A(\text{supp}(\Phi_{\#\#}(c)) = N_A(\text{supp}(\Phi_*(c)))$$

for some uniform constant A .

To get the inverse inclusion, applying the same argument to a coarse inverse $\bar{\Phi}$ we have

$$\bar{\Phi}(\text{supp}(\Phi_*(c))) \subset N_A(\text{supp}(\bar{\Phi}_*\Phi_*(c))) = N_A(\text{supp}(c))$$

where the last equation follows from uniqueness of supports. Now apply Φ to both sides of this equation:

$$\begin{aligned} \text{supp}(\Phi_*(c)) &\subset N_{A'}(\Phi\bar{\Phi}(\text{supp}(\Phi_*(c)))) \\ &\subset N'_A(\Phi(N_A(\text{supp}(c)))) \\ &\subset N_{A''}(\Phi(\text{supp}(c))) \end{aligned}$$

◇

2.3 Application to fiber bundles.

Consider now a fiber bundle $\pi: E \rightarrow B$ with fiber F_x over each $x \in B$. We assume that E, B are UC, ULF simplicial complexes, π is a simplicial map, each fiber $F_x = \pi^{-1}(x)$ is a manifold of dimension n , and for each vertex x the subcomplex F_x is UC, with gauge independent of x . It follows that for each k -simplex σ , the $k + n$ simplices of $\pi^{-1}(\sigma)$ that are not contained in $\pi^{-1}(\partial\sigma)$, intersected with the fiber $F_\sigma = \pi^{-1}(\text{barycenter}(\sigma))$, define a cellular structure on F_σ which is UC, with gauge independent of σ .

Let $d = \dim(B)$, $n = \dim(F)$, $d + n = \dim(E)$.

Make the following assumption about the top dimensional, uniformly finite homology $H_d^{\text{uf}}(B)$:

Top dimensional classes in B separate points: $\exists r > 0$ so that $\forall s > 0 \exists D > 0$ so that for any $x, y \in B$ with $d(x, y) > D$, there is a top dimensional class $c \in H_d^{\text{uf}}(B)$ such that

$$d(\text{supp}(c), x) \leq r \quad \text{and} \quad d(\text{supp}(c), y) > s$$

Example: $T \times \mathbf{R}^n$, where $T = \text{Cayley tree of } F_2$. In the base space T , each bi-infinite line is the support of a top dimensional class, and lines in T clearly separate points.

Example to come: later we shall construct a model space for $\mathcal{MCG}(S_g^1)$, fibering over model space for $\mathcal{MCG}(S_g)$, with fiber \mathbf{H}^2 . The dimension of the base space will equal $4g - 5$, which is the virtual cohomological dimension of $\mathcal{MCG}(S_g)$. We shall prove that the top dimensional classes of uniformly finite homology separate points of $\mathcal{MCG}(S_g)$.

The main result for this section is that every quasi-isometry of the total space E coarsely preserves fibers:

Theorem 14 (Whyte). *Consider the fibration $F \rightarrow E \rightarrow B$ as above, and assume that top dimensional classes in $H_d^{\text{uf}}(B)$ coarsely separate points. For all K, C there exists A such that if $\Phi: E \rightarrow E$ is a K, C quasi-isometry, then for each $x \in B$ there exists $x' \in B'$ such that*

$$d_{\mathcal{H}}(\Phi(F_x), F_{x'}) \leq A$$

Proof. The key observation of the proof is that the support of every top dimensional class in E is saturated by fibers. To be precise: for every top dimensional class $c \in H_{d+n}^{\text{uf}}(E)$, there exists a unique top dimensional class $c' = \pi(c) \in H_d^{\text{uf}}(B)$, such that

$$\text{supp}(c) = \pi^{-1}(\text{supp}(c'))$$

To see why, for each d -simplex $\sigma \subset B$, $\pi^{-1}(\sigma)$ is a manifold with boundary of dimension $d + n$. So, for any class c of dimension $d + n$, if $\text{supp}(c)$ contains some $d + n$ simplex in $\pi^{-1}(\sigma)$, it follows that $\text{supp}(c)$ contains all of $\pi^{-1}(\sigma)$.

The converse is also true: for each top dimensional class $c' \in H_d^{\text{uf}}(B)$ there exists a top dimensional class in E , denoted $c = \pi^{-1}(c') \in H_{d+n}^{\text{uf}}(E)$ such that $\text{supp}(c) = \pi^{-1}(\text{supp}(c'))$: over each simplex $\sigma \subset \text{supp}(c')$, weight all the simplices in $\pi^{-1}(\sigma)$ with the same weight as σ , using a coherent orientation of fibers to choose the sign.

Thus, the projection π induces an isomorphism

$$H_{d+n}^{\text{uf}}(E) \rightarrow H_d^{\text{uf}}(B)$$

so that a $d + n$ -cycle c in E , and the corresponding d -cycle c' in B , are related by

$$\text{supp}(c) = \pi^{-1}(\text{supp}(c'))$$

Now we use the property that supports of top dimensional classes in B separate points. Up to changing constants, it follows that:

Supports of top dimensional classes in E separate fibers: there exists $r > 0$ such that for all $s > 0$ there exists $D > 0$ such that given fibers F_x, F_y with $d_{\mathcal{H}}(F_x, F_y) > D$, there is a top dimensional class $c \in H_{d+n}^{\text{uf}}(E)$ so that $F_x \subset N_r(\text{supp}(c))$ but $F_y \cap N_r(\text{supp}(c)) = \emptyset$.

From this it follows that fibers in E are coarsely respected by a quasi-isometry. Here are the details.

Fix a K, C quasi-isometry $\Phi: E \rightarrow T$ and a fiber F_x , $x \in G$. We want to show that $\Phi(F_x)$ is uniformly Hausdorff close to some fiber $F_{x'}$.

Fix some $R > r$, to be chosen later, and let \mathcal{C}_x denote the collection of classes $c \in H_{d+n}^{\text{uf}}(E)$ such that $F_x \subset N_R(\text{supp}(c))$. From the fact that top dimensional classes in E separate fibers, it follows that F_x has (uniformly) finite Hausdorff distance from the set of points $\xi \in E$ such that $\xi \in N_R(\text{supp}(c))$ for all $c \in \mathcal{C}_x$.

Notation: let $\hat{\mathcal{C}}_x = \{\Phi_{\#}(c) \mid c \in \mathcal{C}_x\}$, and let $\hat{c} = \Phi_{\#}(c)$.

By applying Proposition 13, the Hausdorff distance between $\Phi(\text{supp}(c))$ and $\text{supp}(\hat{c})$ is at most a constant A , for any $c \in \mathcal{C}_x$. Thus for any $\hat{c} \in \hat{\mathcal{C}}_x$ we have

$$\Phi(F_x) \subset N_{R'}(\text{supp}(\hat{c}))$$

where $R' = KR + C + A$. Now we say how large to choose R , namely, so that $R' > r$.

It now follows that $\Phi(F_x)$ has (uniformly) finite Hausdorff distance from the set \mathcal{F} of points $\eta \in E$ such that $\eta \in N_{R'}(\text{supp}(\hat{c}))$ for all $\hat{c} \in \hat{\mathcal{C}}_x$. But the set \mathcal{F} is clearly a union of fibers of E . Pick one fiber $F_{x'}$ in \mathcal{F} . Taking $s = R'$ in the definition of coarse separation of fibers, there is a resulting D . If $F_{y'}$ is a fiber whose distance from $F_{x'}$ is more than D , it follows that $F_{y'}$ is not contained in \mathcal{F} , because that would violate coarse separation of fibers. This shows that the set \mathcal{F} contains $F_{x'}$ and is contained in the D -neighborhood of $F_{x'}$, that is, \mathcal{F} has Hausdorff distance at most D from $F_{x'}$. But \mathcal{F} also has (uniformly) finite Hausdorff distance from $\Phi(F_x)$, and so $\Phi(F_x)$ has (uniformly) finite Hausdorff distance from $F_{x'}$.

This finishes the proof of Theorem 14. \diamond

3 Surface group extensions and Mess subgroups

Let S_g be a closed, oriented surface of genus $g \geq 2$, and let S_g^1 be S_g minus a single base point p . There is a short exact sequence

$$1 \rightarrow \pi_1(S_g) \rightarrow \mathcal{MCG}(S_g^1) \rightarrow \mathcal{MCG}(S_g) \rightarrow 1$$

The homomorphism $\mathcal{MCG}(S_g^1) \rightarrow \mathcal{MCG}(S_g)$ is the map that “fills in the puncture”. The homomorphism $\pi_1(S_g) \rightarrow \mathcal{MCG}(S_g^1)$ is the “push” map, which isotopes the base point p around a loop, at the end of the isotopy defining a map of S_g taking p to itself; then remove the base point to define a mapping class on S_g^1 .

The main theorem of this section is:

Theorem 15. *Every quasi-isometry of $\mathcal{MCG}(S_g^1)$ coarsely preserves the system of cosets of $\pi_1(S_g)$.*

The meaning of this theorem is that for each $K \geq 1$, $C \geq 0$ there exists $A \geq 0$ such that if $\Phi: \mathcal{MCG}(S_g^1) \rightarrow \mathcal{MCG}(S_g^1)$ is a K, C quasi-isometry, then for each coset C of $\pi_1(S_g)$ there exists a coset C' such that $d_{\mathcal{H}}(\Phi(C), C') \leq A$.

The setup of Theorem 15 is to represent the short exact sequence by a fibration

$$\mathbf{H}^2 \rightarrow E \rightarrow B$$

as above, where E is a model space for $\mathcal{MCG}(S_g^1)$, and B is a model space for $\mathcal{MCG}(S_g)$. The theorem can then be translated into geometric terms by saying that every quasi-isometry of E coarsely preserves the fibers.

Whyte’s idea for proving Theorem 15 is to apply Theorem 14, by using Mess subgroups of $\mathcal{MCG}(S_g)$ to provide top dimensional classes in the uniformly finite homology of B that coarsely separate points.

3.1 Dimension of $\mathcal{MCG}(S_g)$.

Given a contractible model space B for a group G , one would expect that the top dimension in which $H_n^{\text{uf}}(B)$ is nontrivial would be $n = \text{vcd}(G)$. So we need the following formula of John Harer:

Theorem 16 ([Har86]).

$$\text{vcd}(\mathcal{MCG}(S_g)) = 4g - 5$$

Proof. We will sketch Harer’s original proof of the upper bound $\text{vcd}(\mathcal{MCG}(S_g)) \leq 4g - 5$, and then we shall give Geoff Mess’ proof of the lower bound $\text{vcd}(\mathcal{MCG}(S_g)) \leq 4g - 5$. Mess’ proof will provide the basic ingredients we need to investigate top dimensional uniformly finite homology classes in a model space for $\mathcal{MCG}(S_g)$.

To prove $\text{vcd}(\mathcal{MCG}(S_g)) \leq 4g - 5$, by using the short exact sequence, in which

$$\text{vcd}(\text{kernel}) = \dim(\mathbf{H}^2) = 2$$

it suffices to prove

$$\text{vcd}(\mathcal{MCG}(S_g^1)) \leq 4g - 3$$

Harer constructs a contractible complex K of dimension $4g - 3$ which is a model space for $\mathcal{MCG}(S_g^1)$: K is the complex of “filling arc systems” of the once punctured surface. Fixing a base point $p \in S_g$, a filling arc system is a system of arcs $\mathcal{A} = \{A_i\}$ whose interiors are pairwise disjoint, whose ends are all located at the base point p , so that for each component C of $S_g - \cup\{A_i\}$, regarding C as the interior of a polygon whose sides are arcs of $\{A_i\}$, the number of sides of C is at least 3. Each filling arc system \mathcal{A} can be refined by adding more arcs until it is triangulated, and the number of such arcs is called the *defect* of \mathcal{A} . The complex K that Harer uses has one cell $C_{\mathcal{A}}$ of dimension d for each isotopy class of filling arc systems \mathcal{A} of defect d , and the boundary of $C_{\mathcal{A}}$ consists of all cells $C_{\mathcal{A}'}$ such that $\mathcal{A} \subset \mathcal{A}'$. Harer used Strebel differentials to prove contradictibility of K , but a purely combinatorial proof was given by Hatcher [Hat91].

Remarks: Harer does *not* directly construct a $4g - 3$ dimensional model space for $\mathcal{MCG}(S_g)$. Thurston, in his three page 1986 preprint “A spine for the Teichmüller space of a closed surface” [Thu86], does construct a model space for $\mathcal{MCG}(S_g)$. With some work, I can prove that the dimension of Thurston’s spine in genus $g = 2$ is indeed equal to $4g - 5 = 8 - 5 = 3$. But I am unable to prove that Thurston’s spine in genus $g \geq 3$ has dimension $4g - 5$, and I think it may be false. Ultimately we will depend on the Eilenberg-Ganea-Wall theorem [Bro82] to obtain an appropriate model space for $\mathcal{MCG}(S_g)$, which is why we need to compute the vcd .

3.2 Mess subgroups

Now we give Geoff Mess' proof of the lower bound: $\text{vcd}(\mathcal{MCG}(S_g)) \geq 4g - 5$.

The proof exhibits a subgroup $M_g < \mathcal{MCG}(S_g)$ which is a Poincaré duality group of dimension $4g - 5$, in fact M_g is the fundamental group of a compact, aspherical $4g - 5$ manifold. The group M_g is called a *Mess subgroup* of $\mathcal{MCG}(S_g)$.

Mess subgroups are constructed by induction on genus.

Base case: Genus 2. With $g = 2$, we have $4g - 5 = 3$, so we need a 3-dimensional subgroup of $\mathcal{MCG}(S_2)$. Take a curve family $\{c_1, c_2, c_3\} \subset S_2$ consisting of three pairwise disjoint, pairwise nonisotopic curves. The Dehn twists about c_1, c_2, c_3 generate a rank 3 free abelian group, and we are done.

Up to the action of $\mathcal{MCG}(S_2)$, there are two orbits of curve families $\{c_1, c_2, c_3\}$, depending on whether or not some curve in the family separates. So, there are two conjugacy classes of Mess subgroups in $\mathcal{MCG}(S_2)$.

Induction step: Let M_{g-1} be a Mess subgroup in $\mathcal{MCG}(S_{g-1})$, and so M_{g-1} is a Poincaré duality group of dimension $4(g - 1) - 5$.

Consider the short exact sequence

$$1 \rightarrow \pi_1(S_{g-1}) \rightarrow \mathcal{MCG}(S_{g-1}^1) \rightarrow \mathcal{MCG}(S_{g-1}) \rightarrow 1$$

Let M'_{g-1} = preimage of M_{g-1} , so we get

$$1 \rightarrow \pi_1(S_{g-1}) \rightarrow M'_{g-1} \rightarrow M_{g-1} \rightarrow 1$$

and it follows that M'_{g-1} is Poincaré duality of dimension $4(g - 1) - 5 + 2$.

Let $S_{g,1}$ be the surface S_g with a hole removed, and with one boundary component. There is a central extension

$$1 \rightarrow \mathbf{Z} \rightarrow \mathcal{MCG}(S_{g,1}) \rightarrow \mathcal{MCG}(S_g^1) \rightarrow 1$$

obtained by collapsing the hole to a puncture; here, the group $\mathcal{MCG}(S_{g,1})$ is defined as the group of homeomorphisms constant on the boundary, modulo isotopies that are stationary on the boundary.

Let M''_{g-1} be the preimage of M'_{g-1} in $\mathcal{MCG}(S_{g,1})$, and we get

$$1 \rightarrow \mathbf{Z} \rightarrow M''_{g-1} \rightarrow M'_{g-1} \rightarrow 1$$

from which it follows that M''_{g-1} is Poincaré duality of dimension $4(g - 1) - 5 + 3$.

Now attach a handle (a one-holed torus) to $S_{g,1}$ to get S_{g+1} , so we get an embedding

$$\mathcal{MCG}(S_{g,1}) \rightarrow \mathcal{MCG}(S_{g+1})$$

Pick a simple closed curve c contained in the handle and not isotopic to the boundary. The Dehn twist τ_c commutes with $\mathcal{MCG}(S_{g,1})$, and in fact the subgroup of $\mathcal{MCG}(S_{g+1})$ generated by τ_c and $\mathcal{MCG}(S_{g,1})$ is isomorphic to the product $\mathcal{MCG}(S_{g,1}) \times \langle \tau_c \rangle$. We can therefore define

$$M_g = M''_{g-1} \times \langle \tau_c \rangle$$

which is a Poincaré duality group of dimension $4(g-1) - 5 + 4 = 4g - 5$ contained in $\mathcal{MCG}(S_g)$.

This finishes Mess' proof that $\text{vcd}(\mathcal{MCG}(S_g)) \geq 4g - 5$. \diamond

Remarks: The construction of M_g is completely determined by the isotopy type of a certain filtration of S_g by subsurfaces. There are only finitely many such isotopy types up to the action of \mathcal{MCG} , and so there are only finitely many conjugacy classes of Mess subgroups. In fact, a little thought shows that there are exactly two conjugacy classes of Mess subgroups, distinguished by whether the original curve system $\{c_1, c_2, c_3\}$ chosen in a genus 2 subsurface with one hole contains a separating curve.

Letting $\text{Stab}(c) < \mathcal{MCG}(S_g)$ be the stabilizer group of the closed curve c picked in the last step, we have

$$M_g \subset \text{Stab}(c)$$

This fact will be significant later on.

3.3 Model spaces and Mess cycles

We will use a small trick: for the moment, we won't actually work with a model space for $\mathcal{MCG}(S_g)$, instead we'll work with a model space for a finite index, torsion free subgroup $\Gamma_g < \mathcal{MCG}(S_g)$. This is OK because the inclusion $\Gamma_g \hookrightarrow \mathcal{MCG}(S_g)$ is a quasi-isometry. Since $\text{vcd}(\mathcal{MCG}(S_g)) = 4g - 5$ it follows that $\text{cd}(\Gamma_g) = 4g - 5$. We can therefore apply the Eilenberg-Ganea-Wall theorem, to obtain a model space B for Γ_g of dimension $4g - 5$.

The reason for this trick is that for a group with torsion such as $\mathcal{MCG}(S_g)$, the construction of a model space of dimension equal to the vcd is problematical.

Next we obtain a model space for E and a fibration $\mathbf{H}^2 \rightarrow E \rightarrow B$ as follows. Start with the canonical \mathbf{H}^2 bundle over Teichmüller space \mathcal{T} , map B to \mathcal{T} by a Γ_g -equivariant map, and pull the bundle back to get E . This space E is then a model space for the canonical $\pi_1(S_g)$ extension of Γ_g , which is quasi-isometric to $\mathcal{MCG}(S_g^1)$. To prove that quasi-isometries of $\mathcal{MCG}(S_g^1)$ coarsely respect cosets of $\pi_1(S_g)$, it suffices to prove the same for the $\pi_1(S_g)$ extension of Γ_g , and for this it suffices to prove that quasi-isometries of E coarsely respect the \mathbf{H}^2 fibers. Applying Theorem 14, it remains to construct top dimensional uniformly finite cycles in B which coarsely separate points.

Given a Mess subgroup $M < \mathcal{MCG}(S_g)$, the intersection $M' = M \cap \Gamma_g$ has finite index in M , and so M' is still Poincaré duality of dimension $4g - 5$. The complex B/M' is

therefore a $K(M', 1)$ space of dimension $4g - 5$. Since M' is a Poincaré duality group, the homology $H_{4g-5}(M')$ is infinite cyclic generated by the fundamental class $[M']$, and this class is represented by a unique $4g - 5$ cycle in B/M' . This cycle lifts to a $4g - 5$ dimensional, uniformly finite cycle in B ; call this a *Mess cycle* in B .

We will prove that the Mess cycles in B separate points.

3.4 Passage to cosets of curve stabilizers

We now pass from Mess cycles to left cosets of Mess subgroups to left cosets of curve stabilizers, as follows. Although \mathcal{MCG} does not act on B , it does quasi-act, which is good enough. The quasi-action of \mathcal{MCG} permutes the Mess cycles. There is a bijection between Mess subgroups and Mess cycles: each Mess subgroup M corresponds to a unique Mess cycle c such that M (coarsely) stabilizes c . If M (coarsely) stabilizes c and if $\Phi \in \mathcal{MCG}(S_g)$ then $\Phi M \Phi^{-1}$ (coarsely) stabilizes $\Phi(c)$.

Pick representatives M_1, \dots, M_k of the finitely many conjugacy classes of Mess subgroups in $\mathcal{MCG}(S_g)$. It follows that, under the quasi-isometry $B \rightarrow \mathcal{MCG}(S_g)$, Mess cycles correspond to left cosets in $\mathcal{MCG}(S_g)$ of M_1, \dots, M_k . So, it suffices to show that left cosets of M_1, \dots, M_k coarsely separate points in $\mathcal{MCG}(S_g)$.

Each Mess subgroup M_i fixes some curve c_i , and so $M_i < \text{Stab}(c_i)$. Thus, each left coset of M_i is contained in a left coset of $\text{Stab}(c_i)$. So, choosing curves c_0, \dots, c_n representing the orbits of simple closed curves, it suffices to prove that the left cosets of the groups $\text{Stab}(c_i)$ coarsely separate points in \mathcal{MCG} .

3.5 New model space

We now switch to a new model space Γ for $\mathcal{MCG}(S_g)$, no longer contractible. We will pass from left cosets of the groups $\text{Stab}(c_i)$ to subsets of the new model space Γ .

Γ is a graph whose vertices are pairs (C, D) where each of C, D is a pairwise disjoint curve system, the systems C, D jointly fill the surface, and each component of $S - (C \cup D)$ is a hexagon. This implies that \mathcal{MCG} acts on the vertex set with finitely many orbits. Since \mathcal{MCG} is finitely generated, and since there are finitely many orbits of vertices, it follows that we can attach edges in an \mathcal{MCG} -equivariant way so that the graph Γ is connected and has finitely many orbits of edges. There's probably some nice scheme for attaching edges, based on low intersection numbers, but it's not necessary. The graph Γ is now quasi-isometric to \mathcal{MCG} . Given a curve c , define Γ_c to be the subgraph of Γ spanned by vertices (C, D) such that $c \in C \cup D$.

3.6 The subgraphs Γ_c coarsely separate points

We can now pass from left cosets of curve stabilizers to the sets Γ_c . Our ultimate goal is to show that the system of subgraphs Γ_c , one for each curve c , coarsely separates points in Γ .

Given vertices (C, D) and (C', D') which are very far from each other, we shall pick a curve c in $C \cup D$ and show that (C', D') is far from Γ_c . This is enough, because (C, D) is contained in Γ_c .

Since (C, D) and (C', D') are very far from each other, there exists $c \in C \cup D$ and $c' \in C' \cup D'$ such that the intersection number $\langle c, c' \rangle$ is very large. Proof: fixing (C, D) , if all such intersection numbers $\langle c, c' \rangle$ are uniformly small, then there is a uniform cardinality to the number of possible (C', D') , so the distance from (C, D) to (C', D') is uniformly bounded.

Consider now any curve system (C_1, D_1) in Γ_c , meaning that (C_1, D_1) contains c . The curve $c \in C_1 \cup D_1$ has very large intersection number with the curve $c' \in C' \cup D'$. It follows that (C_1, D_1) and (C', D') are far from each other. Proof: if (C_1, D_1) and (C', D') are close, there is a uniform bound to the intersection number of a curve in $C_1 \cup D_1$ with a curve in (C', D') .

This completes the proof that quasi-isometries of $\mathcal{MCG}(S_g^1)$ coarsely preserve fibers.

4 Dynamical techniques: extensions of surface groups by pseudo-Anosov homeomorphisms.

In this section we give the proof of Theorem 1, by proving Strong QI-Rigidity of $\mathcal{MCG}(S_g^1)$ in the sense of Section 1. By applying Theorem 15, we are reduced to showing the following:

Fibered QI-rigidity For all $K \geq 1$, $C \geq 0$, $R \geq 0$ there exists $A \geq 0$ such that if Φ is a K, C quasi-isometry of $\mathcal{MCG}(S_g^1)$, and if Φ takes each coset of $\pi_1 S_g$ to within a Hausdorff distance R of some other coset of $\pi_1 S_g$, then there exists $h \in \mathcal{MCG}(S_g^1)$ such that

$$d_{\text{sup}}(\Phi(f), hf) < A \quad \text{for all } f \in \mathcal{MCG}(S_g^1)$$

The methods of proof are very similar to the following result of Farb and myself:

Theorem 17 ([FM02b]). *If F is a Schottky subgroup of $\mathcal{MCG}(S_g)$ with extension*

$$1 \rightarrow \pi_1(S_g) \rightarrow \Gamma_F \rightarrow F \rightarrow 1$$

then the injection $\Gamma_F \rightarrow \text{QI}(\Gamma_F)$ has finite index image. Moreover, if H is a group quasi-isometric to Γ_F then there exists a homomorphism $H \rightarrow \text{QI}(\Gamma_F)$ with finite kernel and finite index image.

In that proof, using a tree as a model space for F , we proved that every quasi-isometry of Γ_F coarsely preserves fibers. Then we used pseudo-Anosov dynamics (as we will here) to prove “Fibered QI-rigidity”, from which Theorem 17 follows.

4.1 Teichmüller space and its canonical \mathbf{H}^2 bundle.

We have already briefly mentioned these objects; here they are in more detail.

The Teichmüller space \mathcal{T}_g of S_g is the space of hyperbolic structures on S_g modulo isotopy, or equivalently the space of conformal structures on S_g modulo isotopy. We also need the Teichmüller space \mathcal{T}_g^1 of S_g^1 , defined similarly using finite area complete hyperbolic structures on S_g^1 , or equivalently conformal structures with a removable singularity at the puncture. It follows that each element of \mathcal{T}_g^1 can be expressed, up to isotopy, as a pair (σ, p) where σ is a hyperbolic structure on S_g representing an element of \mathcal{T}_g , and p is a point in the universal cover $\tilde{\sigma} \approx \mathbf{H}^2$. We therefore obtain a fiber bundle structure

$$\mathbf{H}^2 \rightarrow \mathcal{T}_g^1 \rightarrow \mathcal{T}_g$$

on which the short exact sequence

$$1 \rightarrow \pi_1(S_g) \rightarrow \mathcal{MCG}(S_g^1) \rightarrow \mathcal{MCG}(S_g) \rightarrow 1$$

acts. For each $x \in \mathcal{T}_g$ we use Σ_x to denote the fiber of \mathcal{T}_g^1 over x , and so Σ_x is an isometric copy of \mathbf{H}^2 .

Note that the action of $\mathcal{MCG}(S_g^1)$ on \mathcal{T}_g^1 is *not* cocompact, and so we cannot regard \mathcal{T}_g^1 as a model space of $\mathcal{MCG}(S_g^1)$, and similarly for $\mathcal{MCG}(S_g)$ acting on \mathcal{T}_g . There is, however, a cocompact equivariant spine $Y_g \subset \mathcal{T}_g$ whose inverse image is a cocompact equivariant spine $Y_g^1 \subset \mathcal{T}_g^1$ and we have a fibration

$$\mathbf{H}^2 \rightarrow Y_g^1 \rightarrow Y_g$$

on which the short exact sequence acts. By cocompactness, the Y 's are model spaces for the \mathcal{MCG} 's.

The action of $\mathcal{MCG}(S_g^1)$ on Y_g^1 respects the \mathbf{H}^2 fibers, and the stabilizer of each fiber is $\pi_1 S_g$. It follows that there is a quasi-isometry $\mathcal{MCG}(S_g^1) \rightarrow Y_g^1$ taking cosets of $\pi_1 S_g$ to \mathbf{H}^2 fibers.

We can translate the result of Theorem 15 to the language of Y_g^1 . The translation says: every quasi-isometry $\Phi: Y_g^1 \rightarrow Y_g^1$ coarsely respects fibers, that is, there exists a constant $A \geq 0$ such that for each $x \in Y_g$ there exists $x' \in Y_g$ such that $d_{\mathcal{H}}(\Phi(\Sigma_x), \Sigma_{x'}) \leq A$. Choosing an x' for each x , we obtain an induced map $\phi: Y_g \rightarrow Y_g$ with $\phi(x) = x'$, and ϕ is a quasi-isometry.

This reduces the proof of Fibered QI-rigidity for $\mathcal{MCG}(S_g^1)$ to the analogous statement for quasi-isometries of Y_g^1 : with Φ as above, we must find $h \in \mathcal{MCG}(S_g^1)$ so that the actions of Φ and h on S_g^1 agree to within bounded distance.

\mathbf{H}^2 bundles over lines. Consider bi-infinite, proper paths $\ell: \mathbf{R} \rightarrow \mathcal{T}_g$, with image often in Y_g . The path ℓ is always piecewise smooth and Lipschitz. Let

$$\Sigma_\ell = \pi^{-1}(\ell) \subset \mathcal{T}_g^1$$

so we have an \mathbf{H}^2 bundle over the line ℓ :

$$\mathbf{H}^2 \rightarrow \Sigma_\ell \rightarrow \ell$$

There is a reasonably natural metric on Σ_ℓ , obtained by combining the \mathbf{H}^2 metric on fibers with the metric on the \mathbf{R} factor. There are some choices, but the metric is natural up to quasi-isometry. If ℓ is piecewise geodesic then for the metric on Σ_ℓ we can take the pullback of the metric on \mathcal{T}_g^1 .

Given a quasi-isometry $\Phi: Y_g^1 \rightarrow Y_g^1$ with induced quasi-isometry $\phi: Y_g \rightarrow Y_g$, for any bi-infinite proper path $\ell: \mathbf{R} \rightarrow \mathcal{T}_g$ the map Φ restricts to a fiber respecting quasi-isometry

$$\Sigma_\ell \rightarrow \Sigma_{\phi(\ell)}$$

Also, if $\ell, \ell': \mathbf{R} \rightarrow Y_g$ are “fellow travellers”, then there is an induced $\pi_1 S$ -equivariant map $\Sigma_\ell \rightarrow \Sigma_{\ell'}$ which is a quasi-isometry.

Example: Suppose that ℓ is the axis of a pseudo-Anosov diffeomorphism, or more generally, that ℓ fellow travels such an axis. Thurston’s hyperbolization theorem for fibered 3-manifolds implies that Σ_ℓ is quasi-isometric to \mathbf{H}^3 , and so Σ_ℓ is a Gromov hyperbolic metric space.

Teichmüller geodesics and their singular solv spaces. A *quadratic differential* on S_g is a transverse pair of measured foliations:

$$q = (\mathcal{F}_u, \mathcal{F}_s)$$

Each quadratic differential q determines a singular Euclidean metric, which determines conformal structure with removable singularities, which determines a point $\sigma(q) \in \mathcal{T}_g$. For each $t \in \mathbf{R}$ define

$$q_t = (e^{-t}\mathcal{F}_u, e^t\mathcal{F}_s)$$

The path

$$\gamma_q = \{t \mapsto \sigma(q_t) \mid t \in \mathbf{R}\}$$

in \mathcal{T}_g is the *Teichmüller geodesic* corresponding to q .

Let Σ_q^{SOLV} denote the hyperbolic plane bundle Σ_{γ_q} with the *singular SOLV metric*, defined by

$$e^{-2t} d\mathcal{F}_u^2 + e^{2t} d\mathcal{F}_s^2 + dt^2$$

Fact: if γ_q is cobounded in \mathcal{T}_g , meaning that it is contained in the $\mathcal{MCG}(S_g)$ orbit of some bounded subset B of \mathcal{T}_g , then the identity map $\Sigma_{\gamma_q} \rightarrow \Sigma_q^{\text{SOLV}}$ is a quasi-isometry between the “natural” metric and the singular SOLV metric, with quasi-isometry constants depending only on B .

Pseudo-Anosov homeomorphisms, their axes, and their hidden symmetries.

A homeomorphism $f: S_g \rightarrow S_g$ is *pseudo-Anosov* if there exists a quadratic differential $q_f = (\mathcal{F}_u, \mathcal{F}_s)$ and $\lambda > 1$ such that

$$f(\mathcal{F}_u, \mathcal{F}_s) = (\lambda^{-1}\mathcal{F}_u, \lambda\mathcal{F}_s)$$

It follows that the path γ_{q_f} is invariant under f in \mathcal{T} , and in fact γ_{q_f} is the set of points $\sigma \in \mathcal{T}$ at which $d(f\sigma, \sigma)$ is minimized; we call γ_{q_f} the *axis* of f in \mathcal{T} . Conversely, a mapping class $\Phi \in \mathcal{MCG}(S_g)$ is represented by a pseudo-Anosov homeomorphism only if $d(\Phi\sigma, \sigma)$ has a positive minimum in \mathcal{T} . These facts were proved by Bers [Ber78].

Let Σ_f denote Σ_{q_f} , with a superscript “SOLV” added to the notation if we wish to denote the singular SOLV metric. The group $J_f = \pi_1(S_g) \rtimes_f \mathbf{Z}$ acts by isometries on Σ_f^{SOLV} , that is,

$$J_f < I_f = \text{Isom}(\Sigma_f^{\text{SOLV}})$$

It is possible that J_f is properly contained in I_f . We can think of the elements of $I_f - J_f$ as “hidden symmetries” of f . One possibility for hidden symmetries occurs when f is a proper power. Another possibility occurs when f or some power of f is conjugate to its own inverse. In general, I_f is a virtually cyclic group, and like all such groups there is an epimorphism $I_f \rightarrow C$ whose image C is either infinite cyclic or infinite dihedral, and whose kernel is finite; a nontrivial kernel provides a further source of hidden symmetries of f .

We shall need an alternate description of I_f . There exists a maximal index orbifold subcover $S_g \rightarrow O_f$ such that f descends to a pseudo-Anosov homeomorphism of the orbifold O_f which we shall denote f' . Let $\text{VN}_{f'}$ be the virtual normalizer of f' in $\mathcal{MCG}(O_f)$, consisting of all $g \in \mathcal{MCG}(O_f)$ such that $g^{-1}\langle f' \rangle g \cap \langle f' \rangle$ has finite index in each of the infinite cyclic subgroups $g^{-1}\langle f' \rangle g$ and $\langle f' \rangle$.

Fact 18. *There is a natural extension*

$$1 \rightarrow \pi_1(O_f) \rightarrow I_f \rightarrow \text{VN}_{f'} \rightarrow 1$$

Since the virtual normalizer of a pseudo-Anosov homeomorphism is virtually cyclic, it follows that I_f contains J_f with finite index.

A direct construction can be used to show:

Fact 19. *There exist pseudo-Anosov homeomorphisms with no hidden symmetries, that is, so that $I_f = J_f$.*

For example, consider the fact the dimension of the measured foliation space of S_g is $6g - 6$, and the transition matrix of a train track representative of every pseudo-Anosov homeomorphism f is an $n \times n$ matrix with $n \leq 6g - 6$; it follows that the algebraic degree of the expansion factor $\lambda(f)$ is at most $6g - 6$. One can construct primitive pseudo-Anosov

homeomorphisms f of S_g such that $\lambda(f)$ has maximal algebraic degree $6g - 6$. For such an f , the kernel K of the epimorphism $I_f \rightarrow C$ must be trivial, for if K is nontrivial then f or some power of f commutes with K and so descends through an orbifold covering map $S_g \rightarrow \mathcal{O} = S_g/K$ where the measured foliation space for \mathcal{O} has strictly smaller dimension than for S_g ; it follows that the degree of $\lambda(f)$ is strictly smaller than $6g - 6$.

This still leaves open the possibility that I_f is infinite dihedral, implying that f is conjugate to its own inverse, but a random example will fail to have this property.

4.2 Proof of fibered QI-rigidity

Consider a quasi-isometry $\Phi: Y_g^1 \rightarrow Y_g^1$. As noted in Section 4.1, Φ coarsely respects the fibers of the fibration $Y_g^1 \rightarrow Y_g$, and so Φ induces a quasi-isometry $\phi: Y_g \rightarrow Y_g$. For any bi-infinite, proper path $\ell: \mathbf{R} \rightarrow Y_g$, the quasi-isometry Φ induces a coarse fiber respecting quasi-isometry from Σ_ℓ to $\Sigma_{\phi(\ell)}$. We shall apply various fiber respecting quasi-isometry invariants to the metric spaces Σ_ℓ .

For example, we say that ℓ and its \mathbf{H}^2 bundle Σ_ℓ are *hyperbolic* if Σ_ℓ is a δ -hyperbolic metric space for some $\delta \geq 0$. Since hyperbolicity is a quasi-isometry invariant, we obtain:

Fact 20 (Hyperbolic spaces are preserved). *Every quasi-isometry Φ of $\mathcal{MCG}(S_g^1)$ coarsely respects the hyperbolic spaces Σ_ℓ , and the induced quasi-isometry ϕ of $\mathcal{MCG}(S_g)$ coarsely respects the hyperbolic paths ℓ .*

Much more surprising is that a quasi-isometry Φ coarsely respects the *periodic* hyperbolic spaces. To be precise, a Teichmüller geodesic $\gamma \subset \mathcal{T}_g$, or its \mathbf{H}^2 bundle Σ_γ , is *periodic* if γ is the axis of some pseudo-Anosov mapping class in $\mathcal{MCG}(S_g)$. Equivalently, by Thurston's hyperbolization theorem, Σ_γ is the universal cover of a fibered hyperbolic 3-manifold. A bi-infinite path $\ell: \mathbf{R} \rightarrow Y_g$ is *coarsely periodic* if there exists an axis γ in \mathcal{T} such that ℓ and γ are fellow travellers, meaning that $d(\ell(h(t)), \gamma(t))$ is uniformly bounded where $h: \mathbf{R} \rightarrow \mathbf{R}$ is some quasi-isometry of the real line.

Here is the heart of the matter:

Theorem 21. *Every quasi-isometry of $\mathcal{MCG}(S_g^1)$ coarsely respects the periodic hyperbolic 3-manifolds Σ_γ . To be precise, given: a quasi-isometry $\Phi: Y_g^1 \rightarrow Y_g^1$ inducing a quasi-isometry $\phi: Y_g \rightarrow Y_g$, given a periodic axis $\gamma \subset \mathcal{T}_g$, and given a coarsely periodic path ℓ in Y_g that fellow travels γ , there exists a periodic axis $\gamma' \subset \mathcal{T}_g$ such that $\phi \circ \ell$ fellow travels γ' . Moreover, $\Phi: \Sigma_\ell \rightarrow \Sigma_{\phi(\ell)}$ is a bounded distance from an isometry $H: \Sigma_\gamma^{\text{SOLV}} \rightarrow \Sigma_{\gamma'}^{\text{SOLV}}$.*

The meaning of the final sentence of this theorem is that there is a commutative diagram

of fiber respecting quasi-isometries

$$\begin{array}{ccc} \Sigma_\ell & \xrightarrow{\Phi} & \Sigma_{\phi(\ell)} \\ \downarrow & & \downarrow \\ \Sigma_\gamma^{\text{SOLV}} & \xrightarrow{H} & \Sigma_{\gamma'}^{\text{SOLV}} \end{array}$$

where the vertical arrows represent maps that move points a uniformly bounded distance in \mathcal{T}_g^1 .

In this theorem, the bounds in the conclusion depend only on the quasi-isometry constants of Φ and on the fellow traveller constants for ℓ and γ .

Before proving this theorem we first apply it to:

Proof of QI-rigidity of $\mathcal{MCG}(S_g^1)$.

Fix a quasi-isometry $\Phi: Y_g^1 \rightarrow Y_g^1$.

Consider a pseudo-Anosov homeomorphism $f: S_g \rightarrow S_g$ without hidden symmetries: the group $J_f = \pi_1(S_g) \rtimes_f \mathbf{Z}$ is the entire isometry group of the singular SOLV manifold Σ_f^{SOLV} . It follows that each conjugate gfg^{-1} has no hidden symmetries. By Theorem 21, there is a pseudo-Anosov $f': S_g \rightarrow S_g$ and an isometry $H_f: \Sigma_f^{\text{SOLV}} \rightarrow \Sigma_{f'}^{\text{SOLV}}$ such that Φ takes Σ_f to $\Sigma_{f'}$ by a map which is a bounded distance from H_f . The isometry H_f conjugates $I_f = \text{Isom}(\Sigma_f^{\text{SOLV}})$ to $I_{f'} = \text{Isom}(\Sigma_{f'}^{\text{SOLV}})$, implying that H_f conjugates $\pi_1(\mathcal{O}_f)$ to $\pi_1(\mathcal{O}_{f'})$. However, since f has no hidden symmetries, $\mathcal{O}_f = S_g$, and so $\pi_1(\mathcal{O}_{f'})$ must also equal S_g ; replacing f' if necessary by some root, it follows that f' has no hidden symmetries. The isometry H_f therefore agrees with the action of some automorphism of $\pi_1(S_g)$, which is identified with a mapping class $h_f \in \mathcal{MCG}(S_g^1)$, and conjugation by h_f takes f to f' .

Next we must show that h_f is independent of f . For this we use the well known fact that $\mathcal{MCG}(S_g^1)$ acts faithfully on the circle at infinity S_∞^1 of $\mathbf{H}^2 = \tilde{S}_g$; this fact is the basis of Nielsen theory.

The quasi-isometry Φ also acts on S_∞^1 . To see why, fix a fiber Σ_x and identify this fiber isometrically with \mathbf{H}^2 , so that the boundary of Σ_x is identified with S_∞^1 . Since $\Phi(\Sigma_x)$ is (uniformly) coarsely equivalent to some fiber $\Sigma_{x'}$, and since $\Sigma_{x'}$ is (nonuniformly) coarsely equivalent to Σ_x , the action of Φ induces a self quasi-isometry of Σ_x , thereby inducing a homeomorphism of S_∞^1 .

By construction of h_f , the actions of Φ and of h_f on S_∞^1 agree, that is to say, the action of h_f on S_∞^1 is independent of f . By faithfulness of the action of $\mathcal{MCG}(S_g^1)$ on S_∞^1 , it follows that $h = h_f \in \mathcal{MCG}(S_g^1)$ is independent of f . We have thus constructed the desired mapping class $h \in \mathcal{MCG}(S_g^1)$, and from the construction it is evident that $\Phi: Y_g^1 \rightarrow Y_g^1$ is within bounded distance of the action of h .

◇

4.3 Proof of Theorem 21

This theorem reduces quickly to results from [Mos03] (see also [Bow02]) and from [FM02b], for each of which we will sketch a proof in broad strokes.

Step 1: Hyperbolic lines in \mathcal{T}_g . First we need the following theorem, proved independently by Bowditch and by myself:

Theorem 22 ([Bow02]; [Mos03]). *A line $\ell: \mathbf{R} \rightarrow \mathcal{T}_g$ is hyperbolic if and only if there exists a cobounded Teichmüller geodesic $\gamma: \mathbf{R} \rightarrow \mathcal{T}_g$ that fellow travels ℓ .*

For example, a pseudo-Anosov axis is a cobounded Teichmüller geodesic, forming a countable family. In toto, there are uncountably many cobounded Teichmüller geodesics, making Theorem 21 all the more surprising.

A one minute proof. (Every theorem should have a one minute proof, a five minute proof, a twenty minute proof. . .)

We must construct a quadratic differential $q = (\mathcal{F}_s, \mathcal{F}_u)$.

“Hyperbolicity” means “exponential divergence of geodesics” [Can91]. Ordinarily this applies to geodesic rays passing transversely through spheres, but it also applies to geodesics in Σ_ℓ passing transversely through fibers the $\Sigma_t = \ell^{-1}(t)$ [FM02a]. It follows that every geodesic contained in a fiber Σ_t is stretched exponentially in either the forward or backward direction, as $t \rightarrow +\infty$ or as $t \rightarrow -\infty$. Some geodesics are stretched exponentially in both directions; indeed, this is true of a random geodesic in a fiber. Certain geodesics contained in the fibers Σ_t are stretched exponentially as $t \rightarrow \infty$, but not as $t \rightarrow -\infty$; these geodesics form the unstable foliation \mathcal{F}_u . Certain other geodesics in Σ_t are stretched exponentially as $t \rightarrow -\infty$ but not as $t \rightarrow +\infty$, and these form the stable foliation \mathcal{F}_s .

Taking $q = (\mathcal{F}_u, \mathcal{F}_s)$, a compactness argument shows that ℓ fellow travels the Teichmüller geodesic γ_q . \diamond

Step 2: Periodic hyperbolic lines. The key fact, from which Theorem 21 quickly follows, is:

Theorem 23 ([FM02b]). *Let γ, γ' be cobounded geodesics in \mathcal{T}_g , and suppose that γ is periodic. If there exists a fiber respecting quasi-isometry $\Phi: \Sigma_\gamma^{\text{SOLV}} \rightarrow \Sigma_{\gamma'}^{\text{SOLV}}$ then γ' is periodic and Φ is a bounded distance from an isometry.*

Before sketching the proof, we apply it to:

Proof of Theorem 21. We replace the map $\mathcal{MCG}(S_g^1) \rightarrow \mathcal{MCG}(S_g)$, fibered by cosets of $\pi_1 S_g$, with the map $Y_g^1 \rightarrow Y_g$, fibered by copies of \mathbf{H}^2 . Let $\Phi: Y_g^1 \rightarrow Y_g^1$ be a quasi-isometry, inducing a quasi-isometry $\phi: Y_g \rightarrow Y_g$. Let $\gamma \subset \mathcal{T}_g$ be a periodic axis fellow travelling a coarsely periodic path ℓ . We obtain a fiber respecting quasi-isometry $\Sigma_\ell \rightarrow \Sigma_\gamma^{\text{SOLV}}$.

The quotient $\Sigma_\gamma^{\text{SOLV}}$ modulo its isometry group is a hyperbolic 3-orbifold, by Thurston's hyperbolization theorem, and so $\Sigma_\gamma^{\text{SOLV}}$ is quasi-isometric to \mathbf{H}^3 . It follows that Σ_ℓ is a hyperbolic metric space, that is, ℓ is a hyperbolic line. By Fact 20, $\phi(\ell)$ is a hyperbolic line and $\Sigma_{\phi(\ell)}$ is a hyperbolic metric space. Applying Theorem 22, there is a cobounded geodesic γ' in \mathcal{T} that fellow travels $\phi(\ell)$. By combining the fiber respecting quasi-isometries $\Sigma_\gamma^{\text{SOLV}} \rightarrow \Sigma_\ell \rightarrow \Sigma_{\phi(\ell)} \rightarrow \Sigma_{\gamma'}^{\text{SOLV}}$, we obtain a fiber respecting quasi-isometry $\Sigma_\gamma^{\text{SOLV}} \rightarrow \Sigma_{\gamma'}^{\text{SOLV}}$. Applying Theorem 23, γ' is periodic the latter quasi-isometry is a bounded distance from an isometry. \diamond

Step 3: Proof of Theorem 23. Farb and I gave a proof that uses Thurston's hyperbolization theorem for fibered 3-manifolds [Ota96] together with Rich Schwartz' geodesic pattern rigidity theorem [Sch97]. It would be extremely nice to have a proof which uses only pseudo-Anosov dynamics, but I still don't know how to do this. Here is a broad sketch of the proof of Farb and myself taken from [FM02b].

Given a cobounded Teichmüller geodesic γ , define $\text{QI}_f(\Sigma_\gamma^{\text{SOLV}})$ to be the group of “fiber respecting quasi-isometries” of $\Sigma_\gamma^{\text{SOLV}}$. We have an injection $\text{Isom}(\Sigma_\gamma^{\text{SOLV}}) = I_\gamma \hookrightarrow \text{QI}_f(\Sigma_\gamma^{\text{SOLV}})$, and the question arises whether there is anything else in $\text{QI}_f(\Sigma_\gamma^{\text{SOLV}})$.

First we prove, when γ is a periodic geodesic, that the injection $I_\gamma \rightarrow \text{QI}_f(\Sigma_\gamma^{\text{SOLV}})$ is an isomorphism, that is, every self quasi-isometry of $\Sigma_\gamma^{\text{SOLV}}$ that coarsely respects fibers is a bounded distance from an isometry.

The singular lines of $\Sigma_\gamma^{\text{SOLV}}$ form a collection of singular SOLV geodesics intersecting the fibers at right angles; let Ω denote this collection of geodesics. If Φ is a fiber respecting quasi-isometry of $\Sigma_\gamma^{\text{SOLV}}$, then Φ coarsely respects leaves of f^s and f^u as noted earlier, and in fact Φ coarsely respects the suspensions of these leaves. It follows that Φ coarsely respects Ω , because the singular lines in Ω are precisely the sets which, coarsely, are intersections of three or more suspensions of leaves of f^s (or of f^u) whose pairwise intersections are unbounded. We may then move values of Φ by a bounded amount so that Φ is a homeomorphism that strictly respects leaves of f^s , leaves of f^u , and Ω .

By Thurston's hyperbolization theorem, there is an I_γ -equivariant \mathbf{H}^3 metric on $\Sigma_\gamma^{\text{SOLV}}$. The lines in Ω can be straightened to hyperbolic geodesics, which are evidently invariant under I_γ . This is exactly the setup of Schwartz' theorem, whose conclusion is that the group of quasi-isometries of \mathbf{H}^3 that coarsely respects Ω contains I_γ with finite index, that is, $\text{QI}_f(\Sigma_\gamma^{\text{SOLV}})$ contains I_γ with finite index. But then an easy argument shows that I_γ must actually be all of $\text{QI}_f(\Sigma_\gamma^{\text{SOLV}})$.

Using the isomorphism $I_\gamma \rightarrow \text{QI}_f(\Sigma_\gamma^{\text{SOLV}})$ and the coarse fiber respecting quasi-isometry $\Phi: \Sigma_\gamma^{\text{SOLV}} \rightarrow \Sigma_{\gamma'}^{\text{SOLV}}$, we now show that Φ is a bounded distance from an isometry. As above, we may first move values of Φ by a bounded amount so that Φ is a homeomorphism that respects the fibers and the stable and unstable foliations. Let Σ_x denote a fiber of $\Sigma_\gamma^{\text{SOLV}}$ whose image under Φ is a fiber $\Sigma_{x'}$ of $\Sigma_{\gamma'}^{\text{SOLV}}$. Let I_x be the subgroup of $\text{Isom}(\Sigma_x)$ that

preserves the stable foliation and the unstable foliation, and similarly for $I_{x'}$. The conjugate action $\Phi^{-1} \circ I_{x'} \circ \Phi$ is an action on $\Sigma_\gamma^{\text{SOLV}}$ by quasi-isometries that preserves each fiber. By the computation of $\text{QI}_f(\Sigma_\gamma^{\text{SOLV}})$ just given, we obtain $\Phi^{-1} \circ I_{x'} \circ \Phi \subset I_x$, and in particular $\Phi^{-1} \circ I_{x'} \circ \Phi$ preserves the invariant measures on the stable and unstable foliations of Σ_x . Conjugating back now to $\Sigma_{x'}$, it follows that $I_{x'}$ preserves two sets of invariant measures on the stable and unstable foliations of $\Sigma_{x'}$: the ones coming from the singular SOLV structure on $\Sigma_{\gamma'}^{\text{SOLV}}$, and the ones pushed forward via Φ . But the stable and unstable foliations on $\Sigma_{x'}/I_{x'}$ are uniquely ergodic: this follows from a theorem of Masur [Mas80], which says that the stable and unstable foliations associated to a cobounded geodesic in Teichmüller space are uniquely ergodic. Up to rescaling, therefore, the map that Φ induces from Σ_x to $\Sigma_{x'}$ is an isometry; and the rescaling may be ignored by moving the point x' up or down. But this immediately implies that $\Sigma_\gamma^{\text{SOLV}}$ and $\Sigma_{\gamma'}^{\text{SOLV}}$ are isometric, by an isometry that agrees with Φ on $\Sigma_x \rightarrow \Sigma_{x'}$ and that moves other fibers up or down by uniformly bounded adjustments.

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